

# COLOQUIO DE ANÁLISIS Y FÍSICA–MATEMÁTICA

Organizers: Dr. Rafael del Río y Dr. Ricardo Weder

## **ESSENTIAL SPECTRUM AND EXPONENTIAL ESTIMATES OF EIGENFUNCTIONS OF PARTIAL DIFFERENTIAL OPERATORS. APPLICATIONS TO SCHRÖDINGER AND DIRAC OPERATORS**

**DR. VLADIMIR RABINOVICH**

Instituto Politécnico Nacional

### **Abstract**

The main aims of the talk are the presentation of results on:

1. The location of the essential spectra of elliptic systems of partial differential operators on  $\mathbb{R}^n$ .
2. Exponential estimates of their eigenfunctions.

Our approach to these problems is based on the theory of limit operators (see, V. Rabinovich, S. Roch, B. Silberman, *Limit operator method and its applications in operator theory*. In *Operator Theory: Advances and Applications*, Vol. 164, 2004, Birkhauser Verlag).

As applications we consider the location of essential spectra of Schrödinger operators, in particular, of multiparticle Schrödinger operators, and exponential estimates of the eigenfunctions of the discrete spectra.

The similar questions will be considered for Dirac operators.

*12 de noviembre de 2009.*



## **SISTEMAS HAMILTONIANOS DISCRETOS Y FINITOS CON VISTAS AL MANEJO DE SEÑALES FINITAS E IMÁGENES PIXELADAS POR MEDIOS ÓPTICOS**

**DR. KURT BERNARDO WOLF**

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### **Resumen**

Los sistemas que se acomodan en grupos compactos, como el oscilador armónico en el grupo  $SU(2)$ , tienen realizaciones en óptica geométrica, ondulatoria y en un modelo óptico donde las posiciones y momentos son discretos y finitos –los estados del sistema son señales finitas de  $N$  puntos. A este modelo corresponde un espacio fase que es una esfera, y una función de distribución de Wigner que nos permite "ver" las señales sobre ella. Presentamos una clasificación completa de las aberraciones de señales finitas en correspondencia con su listado en óptica geométrica. En dos dimensiones tenemos imágenes cuadradas compuestas por  $N \times N$  píxeles, que pueden ser sujetas a rotación, giración, o transformación de pixelado cartesiano a pixelado circular.

*18 de junio de 2009.*



## **RIEMANN–HILBERT PROBLEM APPROACH TO LONG–TIME ASYMPTOTICS FOR NONLINEAR INTEGRABLE PROBLEMS, II: THE CAMASSA–HOLM EQUATION**

**Anne Boutet de Monvel**

Universidad de París

## Abstract

We develop a Riemann–Hilbert approach for the Camassa–Holm equation  $u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$  on the line  $-\infty < x < \infty$ . This approach allows us to study the long–time behavior of solutions of initial value problems for this equation. Regions of the half plane  $-\infty < x < \infty, t \geq 0$ , with different long–time behavior are specified.

We obtain analogous results for initial–boundary value problems for the Camassa–Holm equation on the half–line  $x > 0$  with time decaying boundary conditions at  $x = 0$ . Regions of the quarter plane  $x \geq 0, t \geq 0$ , with different long–time behavior are specified.

Work in collaboration with Dmitry Shepelsky.

27 de mayo de 2008.



## RIEMANN–HILBERT PROBLEM APPROACH TO LONG–TIME ASYMPTOTICS FOR NONLINEAR INTEGRABLE PROBLEMS, I: THE NONLINEAR SCHRÖDINGER EQUATION

**Anne Boutet de Monvel**

Universidad de París

## Abstract

We develop a Riemann–Hilbert approach to the focusing nonlinear Schrödinger equation  $iq_t + q_{xx} + 2|q|^2 q = 0$ , on the first quarter plane  $x > 0, t > 0$  for fast decaying initial data  $q_0(x)$  and time–periodic boundary data  $g_0(t)$ . This approach allows us to study the long–time behavior of solutions of this initial boundary value problem. Regions of the quarter plane  $x \geq 0, t \geq 0$  with different long–time behavior are specified.

However, this approach requires certain a priori information about the long–time behavior of the boundary values which do not involved in the formulation of a well–posed initial boundary value problem (for example, the Neumann boundary values for the Dirichlet problem). Various possibilities for this are discussed and illustrated numerically in the case of one–frequency boundary data.

Work in collaboration with Alexander Its, Vladimir Kotlyarov, Dmitry Shepelsky, and Chunxiong Zheng (numerics).

20 de mayo de 2008.



## ON THE INVERSE SCATTERING PROBLEM FOR THE HELMHOLTZ EQUATION ON THE REAL LINE

**Dra. Ingrid Belitiță**

Institute of Mathematics “Simion Stoilow” of the Romanian Academy

## Abstract

We consider the inverse scattering problem for the equation

$$u'' + \frac{k^2}{c^2} u = 0,$$

on  $\mathfrak{R}$ , where  $c$  is a real measurable function satisfying  $c(x) \geq c_m > 0$  and  $|c(x) - 1| \leq C \langle x \rangle^{-1-\delta}$ ,  $\delta > 0$ .

The situation is very different from the case of the Schrödinger operators, since the Jost functions and the scattering coefficients do not have simple behaviour at infinity in the energy variable. We analyze this behaviour and show that the analysis of analytic functions (but not necessarily of Hardy classes) on the half-spaces plays an important role in proving the uniqueness in the inverse scattering problem.

14 de mayo de 2008.



## LOCAL SMOOTHING FOR THE BACKSCATTERING TRANSFORM FOR SCHRÖDINGER OPERATORS

**Dra. Ingrid Belitiță**

Institute of Mathematics "Simion Stoilow" of the Romanian Academy

### Abstract

We consider the inverse backscattering problem for Schrödinger operators  $H_v = -\Delta + v$ ,  $v \in L_{comp}^\infty(\mathfrak{R}^n; \mathfrak{R})$ , in odd dimensions  $n \geq 3$ . The backscattering transform  $Bv$  of the potential  $v$  is, up to a smooth function, the real part of the inverse Fourier transform of the backscattering part of the scattering matrix. The mapping  $L_{comp}^\infty(\mathfrak{R}^n) \ni v \mapsto Bv \in D'(\mathfrak{R}^n)$  is entire analytic, and we write

$$Bv = \sum_1^\infty B_N v,$$

where  $B_N v$  is the  $N$ :th order term in the power series expansion of  $B$  at  $v = 0$ . Here  $B_1 v = v$ . We shall present estimates for  $B_N v$  in Sobolev spaces  $H_{(s)}(\mathfrak{R}^n)$  and show that  $v \mapsto Bv$  extends to an entire analytic mapping on  $H_{(s)}(\mathfrak{R}^n) \cap \mathcal{E}'(\mathfrak{R}^n)$  with values in  $H_{(s),comp}(\mathfrak{R}^n)$ , when  $s \geq (n-3)/2$ . We show moreover that, when  $s > (n-3)/2$  and  $v \in H_{(s)}(\mathfrak{R}^n) \cap \mathcal{E}'(\mathfrak{R}^n)$ , the regularity of  $B_N v$  increases with  $N$  and  $v - Bv$  is locally of class  $H_{(s+a)}(\mathfrak{R}^n)$ , where  $0 \leq a < 1, a < s - (n-3)/2$ .

*This talk reports on joint work with Anders Melin (Lund University).*

6 de mayo de 2008.



## RECUPERACIÓN DE LA FUNCIÓN DE GREEN EN ELASTICIDAD DINÁMICA A PARTIR DE CORRELACIONES DEL RUIDO

**Dr. Francisco J. Sánchez-Sesma**

Instituto de Ingeniería-UNAM

26 de febrero de 2008.



## SPECTRAL THEORY OF QUANTUM GRAPHS

**Dr. Pavel Kurasov**

Lund Institute of Technology

## Abstract

Under quantum graphs we understand differential operators on metric graphs. Such operators are determined by: 1) a metric graph; 2) differential expressions on the edges and 3) boundary conditions at the vertices.

These operators form a special class possessing features of both ordinary and partial differential operators. Quantum graphs can be used to model narrow wave guides, quantum wires, carbon nano-structures, etcetera, and serve as toy models in the theory of dynamical systems and quantum mechanics. The spectral theory of quantum graphs is a rapidly developing area of research in modern mathematical physics. This lecture course is devoted to modern developments in the area and will be suitable as an introduction into this research area.

Three types of quantum graphs will attract our attention:

- ✦ compact finite graphs;
- ✦ noncompact graphs obtained by attaching few infinite edges;
- ✦ self-similar graphs.

It has been shown that the spectrum of finite compact graphs is pure discrete and in order to calculate it one way view these graphs as quantum billiards. In this way it is possible to establish a trace formula connecting the spectrum of a quantum graph and the set of periodic orbits of the underlying metric graph. Moreover, in this case the inverse spectral problem will be discussed.

The spectrum of non-compact graphs contain branches of absolutely continuous spectrum which gives rise to the study of direct and inverse scattering problems. Such models are important for the calculation of transition probabilities in quantum networks.

Self-similar graphs serve as models of fractals and attract attention due to unusual spectral properties. The preliminary plan for the course looks as follows:

1. Quantum graphs as coupled ordinary differential operators: definition, self-adjoint vertex conditions, elementary spectral properties and counterexamples.
2. Trace formula for graph Laplacians: spectral and topological invariants. Can one hear the shape of a graph?
3. On inverse scattering problems for noncompact graphs: uniqueness theorems and explanation of counterexamples.
4. Graphs with rationally dependent lengths of edges and self-similar graphs (fractal models).

*13, 15, 20 y 22 de marzo de 2007.*



## PROBABILIDADES CONDICIONALES Y COLAPSO EN LAS MEDICIONES CUÁNTICAS

**Dr. Roberto Laura**

Universidad Nacional de Rosario

### Resumen

Incluyendo el sistema y el instrumento de medición en la descripción cuántica del proceso de medición, y usando el concepto de probabilidades condicionales, es posible deducir el operador estadístico del sistema después de la medición. Este operador estadístico determina la distribución de probabilidad para todas las mediciones posteriores, y en general no coincide con el que se obtiene usando el postulado del colapso.

*23 de enero de 2007.*



## EXPLICIT SOLUTIONS TO THE KORTEWEG-DE VRIES EQUATION ON THE HALF LINE

**Dr. Tuncay Aktosun**

University of Texas at Arlington

## Abstract

We analyze the Korteweg–de Vries equation on the half line  $u_t + \eta u_x - 6uu_x + u_{xxx} = 0$ , where  $\eta$  is a nonnegative constant,  $x \geq 0$ , and  $t \geq 0$ . We obtain certain explicit solutions in terms of elementary function; such solutions contain those that are global in time (i.e. valid for all  $t \in [0, +\infty)$ ) and also those local in time (i.e. valid for  $t \in [0, \tau)$  for some positive  $\tau$ ). The initial values of these solutions are associated with rational scattering data for the related Schrödinger equation.

*15 de noviembre de 2006.*



## A BALANCE OF KINETIC AND POTENTIAL ENERGIES IN THE SEMICLASSICAL LIMIT

**Dr. Dimitri Yafaev**

Université de Rennes–1

## Abstract

One considers a bound state of a quantum particle in a potential well. The goal is to find a limit of the kinetic (or potential) energy as the Planck constant tends to zero.

*4 de mayo de 2006.*



## SCATTERING THEORY FOR NON–LINEAR SCHRÖDINGER EQUATIONS

**Dr. Nakao Hayashi**

Osaka State University

*23 de enero de 2006.*



## THE SAARI PROBLEM FOR MECHANICAL SYSTEMS WITH SYMMETRY

**Dr. Jeffrey Lawson**

Western Carolina University

## Abstract

Saari's Conjecture states that an N–body system (N point particles mutually attracted by Newtonian potentials) has a constant moment of inertia if and only if the system is in relative equilibrium. J. Marsden suggested that Saari's Conjecture may be extended to more general dynamical systems with symmetry, but in recent literature there have appeared non–Newtonian N–body arrangements that would contradict a more general Saari's Conjecture.

A. Hernández, J. Lawson, and Marsden found a novel counterexample to Marsden's conjecture in the case of a free rigid body in 3 dimensions. This counterexample motivated an appropriate refinement of the conjecture by interpreting the moment of inertia using the locked inertia tensor. We can now claim that mechanics on any free Lie group constitutes a solution to the "Saari Problem".

In recent work with C. Stoica, we consider a simple mechanical system with symmetry using Palais slices in a neighborhood of a point of trivial isotropy (with respect to the group action). The relationship between the constancy of the locked inertia tensor and relative equilibria allow us to contribute more solutions to the Saari Problem, including the 3– and 4–body problems in 3 dimensions.

*6 de enero de 2006.*



## EXISTENCIA CASI-GLOBAL DE SOLUCIONES A ECUACIONES DE KLEIN-GORDON HAMILTONIANAS, SEMI-LINEALES, CON DATOS DE CAUCHY PEQUEÑOS EN VARIEDADES DE ZOLL

**Dr. Benoît Grébert**  
Université de Nantes

### Resumen

En este coloquio se discutirá la existencia casi global de soluciones a ecuaciones de Klein–Gordon en variedades de Zoll (por ejemplo esferas de dimensión arbitraria). Las demostraciones están basadas en métodos de forma normal de Birkhoff y en la distribución de los autovalores del Laplaciano perturbado por un potencial en variedades de Zoll.

14 de diciembre de 2005.



## INTRODUCTION TO CALDERÓN'S PROBLEM AND OUTLINE OF THE NEW COMPLEX ANALYSIS METHOD

**Dr. Lassi Päivärinta**  
Helsinki University

### Abstract

The basic problem of *Electrical Impedance Tomography* is to determine the conductivity of a body by measuring voltages and currents on its surface. It has been proposed as a valuable diagnostic tool especially for detecting breast cancer and for monitoring heart and lungs.

Is this really possible? Can one formulate the problem in mathematical terms?

The answers to both questions are positive! In 1980 A.P. Calderón put the question in to the language of mathematics by formulating it in the following way:

Suppose  $\Omega \subset \mathbb{R}^n$  is smooth and bounded,  $\sigma \in L^\infty(\Omega)$  is bounded away from zero, and that  $u \in H^1(\Omega)$  is the weak solution of

$$\begin{aligned} \operatorname{div}(\sigma \nabla u) &= 0 \\ u|_{\partial\Omega} &= f \end{aligned}$$

where  $f \in H^{1/2}(\partial\Omega)$ . Define the Dirichlet–to–Neumann map:  $\Lambda_\sigma: H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$  by

$$\Lambda_\sigma(f) = \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}$$

where  $\nu$  is the unit outer normal to  $\partial\Omega$ . Does  $\Lambda_\sigma$  uniquely determine  $\sigma$ ? In physical terms  $u|_{\partial\Omega} =$

$f$  represents the electric potential i.e. voltage and  $\sigma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}$  the electric current at the boundary of  $\Omega$ .

In the four lectures we will discuss the history of this problem and introduce some new techniques of complex analysis to attack the problem. We demonstrate how this problem can completely and constructively be solved in two dimensions in its original form. In particular, we prove uniqueness of the inverse problem. The methods involve quasiconformal techniques, harmonic analysis, Beals–Coifman theory and finally some topological arguments. The work is a joint study with K. Astala from Helsinki.

16 de noviembre de 2005.



## GEL'FAND INVERSE BOUNDARY SPECTRAL PROBLEM

- ✧ Gelfand inverse boundary spectral problem for a Schrödinger operator on a Riemannian manifold: Analytical and geometrical aspects of the boundary control method. Part I.
- ✧ Gelfand inverse boundary spectral problem for a Schrödinger operator on a Riemannian manifold: Analytical and geometrical aspects of the boundary control method Part II.
- ✧ Inverse boundary value problem for the Dirac and Maxwell equations on a manifold.
- ✧ Geometric convergence and stability in the Gelfand inverse problem.

**Dr. Slava Kurylev**

Loughborough University

*20, 21, 26 y 28 de abril de 2005.*



## TIME-DOMAIN AND FREQUENCY-DOMAIN METHODS FOR INVERSE SCHRÖDINGER SCATTERING

**Dr. Tuncay Aktosun**

Mississippi State University

### Abstract

We review various methods to recover the potential of the one-dimensional Schrödinger equation when that potential is real valued, integrable, has a finite first moment, has no bound states, and has support on the positive axis. We compare the time-domain and frequency-domain methods and explore the connections among them.

*8 de diciembre de 2004.*



## HIGH ENERGY AND SMOOTHNESS ASYMPTOTIC EXPANSION OF THE SCATTERING AMPLITUDE

**Dr. Dimitri Yafaev**

Université de Rennes-1

### Abstract

We find an explicit expression for the kernel of the scattering matrix for the Schrödinger operator containing at high energies all terms of power order. It turns out that the same expression gives a complete description of the diagonal singularities of the kernel in the angular variables. The formula obtained is in some sense universal since it applies both to short -and long- range electric as well as magnetic potentials.

*27 de octubre de 2004.*



## TWO SHARP RESULTS IN SPECTRAL THEORY OF SCHRÖDINGER OPERATORS

**Dr. Alexander Kiselev**

Wisconsin University

### Abstract

We will discuss two results on the spectrum of one-dimensional Schrödinger and Stark operators. First, we prove that for any potential decaying at the Coulomb rate, the singular continuous spectrum is empty, and so modified wave operators are asymptotically complete. Second, we show that Stark operators with rough periodic perturbations can have purely singular spectrum if the perturbation is in  $H^{-1/2}$ . On the other hand, it is known that if perturbation is in  $H_s$  with  $s > 1/2$ , the absolutely continuous spectrum fills the whole real axis.

*11 de agosto de 2004.*



## **CAN YOU DETERMINE THE “SHAPE” OF A VOCAL TRACT**

**Dr. Tuncay Aktosun**

Mississippi State University

### **Abstract**

In this joint work with Roy Pike of King's College of London and Barbara Forbes of Phonologica, London, we analyze the inverse problem of determining the scaled curvature of a human vocal tract. Mathematically, this problem is equivalent to finding the potential of the radial Schrödinger equation when the spatial derivative of the Jost solution is known at the origin.

*19 de mayo de 2004.*