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Semi-classical Analysis and the Stability of Matter

In 1967, Dyson and Lenard [DL] proved that (neutral) matter composed of electrons and nuclei interacting via Coulomb forces is stable in the sense that the ground state energy is proportional to the number of constituents. Later, Lieb and Thirring [LT] found a different and simple proof of this stability based on Thomas-Fermi theory [LS]. Since then, these seminal papers have sparked exciting research activities in mathematical physics and left many open problems to challenge future generations.

One fundamental question that we discuss in these lectures is the ground state energy, $E(N)$, of neutral atoms composed of N electrons. An asymptotic expansion of the form

$$E(N) = a_{\mathrm{TF}} N^{7/3} + \tfrac{1}{4} N^2 + o(N^2)$$

was proved by Ivrii and Sigal [IS] in 1993 (see also Siedentop and Weikard [SW]). The first term in this series, $a_{\mathrm{TF}} N^{7/3}$ with $a_{\mathrm{TF}} \approx -0.485$, is the Thomas-Fermi energy term that was already found in 1927 and rigorously proved by Lieb and Simon [LS]. The second term is called the Scott correction which was discovered in 1955.

A key ingredient in the proof of the expansion for $E(N)$ as $N \to \infty$ is a semi-classical analysis ($h \downarrow 0$) of a one-particle Schrödinger operator $H = -h^2 \Delta + V$. Here, the semi-classical parameter, h, in front of the Laplace operator $-\Delta$ is related to the number of electrons, N, through $h = N^{-1/3}$ and V is the Thomas-Fermi potential.

Jointly with Solovej [SS] we developed an improved semi-classical calculus of coherent states. More recently, we have extended this to matter composed of pseudo-relativistic electrons and proved the relativistic Scott correction [SSS] (see also Frank, Siedentop, and Warzel [FSW]).

In the lectures we first discuss some basic tools such as the Lieb-Thirring inequality and Thomas-Fermi theory. Then we introduce the calculus of coherent states of [SS] and apply this to prove the main semi-classical result on the sum of negative energies of the above Schrödinger operator H.

References

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