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Adiabatic Perturbation Theory in Quantum Dynamics

Separation of scales plays a fundamental role in the understanding of the dynamical behavior of complex systems in physics and other natural sciences. It is often possible to derive simple laws for certain slow variables from the underlying fast dynamics whenever the scales are well separated. The prototypic example are molecules, i.e. systems consisting of two types of particles with very different masses. Electrons are lighter than nuclei by at least a factor of $2 \cdot 10^3$, depending on the type of nucleus. Therefore, assuming equal distribution of kinetic energies inside a molecule, the electrons are moving at least 50 times faster than the nuclei. The effective dynamics for the slow degrees of freedom, i.e. for the nuclei, is known as the Born-Oppenheimer approximation and it is of extraordinary importance for understanding molecular dynamics. Roughly speaking, in the Born-Oppenheimer approximation the nuclei evolve in an effective potential generated by one energy level of the electrons, while the state of the electrons instantaneously adjusts to an eigenstate corresponding to the momentary configuration of the nuclei. The phenomenon that fast degrees of freedom become slaved by slow degrees of freedom which in turn evolve autonomously is called adiabatic decoupling.

In this series of lectures I will give an overview of different aspects and applications of adiabatic perturbation theory for quantum systems, focussing on recent developments. One aim is to illustrate the various very different realizations of adiabatic decoupling in physics and to explain their common mathematical structure.

Lecture 1: The classical time-adiabatic theorem

The classic adiabatic theorem of quantum mechanics going back to Born and Fock and to Kato (see [Te_2] for references) is concerned with quantum systems whose Hamiltonian depends explicitly and slowly on time. This theorem has a number of mathematical generalizations and an important aspect are geometric (or Berry) phases. As a nontrivial recent physical application I will discuss the Piezo effect in crystalline solids, where the slow deformation of a periodic potential gives rise to a geometric current [PSpT].

Lecture 2: Space adiabatic perturbation theory
In most interesting systems showing a separation of scales the Hamiltonian is time-independent. Instead, the slow variation of some degrees of freedom is of dynamical origin, e.g. because of their large masses or because of slowly varying external potentials. In a suitable representation the Hamiltonian of such a system has the form of an pseudo-differential operator with an operator valued symbol. There is a general mathematical formalism for proving adiabatic decoupling for systems guided by Hamiltonians of that form called adiabatic perturbation theory, c.f. [MaSo, NeSo, PST, Te_2].

In this lecture I will explain the general formalism of space adiabatic perturbation theory leading to effective Hamiltonians for the slow degrees of freedoms in such systems. The perturbation parameter $\epsilon \ll 1$ is a dimensionless ratio of two temporal or spatial scales. As a technically simple example, but having all the interesting structure, I discuss the semiclassical limit of the Dirac equation [PST].

Lecture 3: Constrained quantum systems
As a rather recent and highly nontrivial application I discuss strongly confined quantum systems and the limit of holonomic constraints to submanifolds. Adiabatic perturbation theory allows to extend results of the type [FrHe] to the regime of large tangential kinetic energies, where the effective equations for the constraint system have a much richer structure [TeWa].

Lecture 4: Exponential estimates and non-adiabatic transition histories
The adiabatic approximations discussed until now ignore non-adiabatic transitions between almost invariant subspaces. Under suitable analyticity assumptions these transitions are of order $\exp(-c/\epsilon)$ for some $c > 0$ and thus do not appear in any order of perturbation theory in ϵ. However, the transitions have important physical consequences and it is an interesting task to go beyond all orders of perturbation theory and to quantify them. This leads to so called Landau-Zener formulas. In this lecture I present the traditional approach [JoPf] and a more recent approach [BeTe] on Landau Zener formulas and non-adiabatic transition histories.

Lecture 5: Extensions: adiabatic decoupling for systems without spectral gap or complex eigenvalues
In all situations discussed up to now a crucial ingredient for applying adia-

batic perturbation theory was a uniform spectral gap. In this lecture I discuss adiabatic theorems and applications where this condition is violated. This violation comes in two flavors: as eigenvalues at the bottom of continuous spectrum [AvEl, Te_1] or as resonances, i.e. complex eigenvalues after suitable analytic deformations [AbFr]. As an application I discuss the derivation of quantum mechanics from a model of non-relativistic QED [TeTe].

In all lectures I will assume familiarity with basic mathematical concepts in quantum mechanics. More advanced concepts like pseudo-differential operators with operator valued symbols and the corresponding calculus are introduced along the way.

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